OBSERVATIONS

RELATIVE TO

THE FORM OF THE

ARBITRARY CONSTANT QUANTITIES

THAT OCCUR IN THE INTEGRATION OF

CERTAIN DIFFERENTIAL EQUATIONS;

AND, ALSO, IN THE INTEGRATION OF A

CERTAIN EQUATION OF FINITE DIFFERENCES.

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IT is well known that the integral of a differential equation of any order will contain as many constant arbitrary quantities as the order contains units, and it is generally assumed that they are entirely arbitrary, as well in form as in quantity. But making them arbitrary as to form, oftentimes makes them less comprehensive than they would otherwise be.

Let $\varphi(x, y, a, c_1, c_2, \&c.) = o$ be the complete integral of $\varphi'(x, y, a, dx, dy, d^2y\ \&c.) = o$
then c_1, c_2, &c. being arbitrary as to form, it may happen that although the former equation be exact when a is any assigned quantity, yet if its limit be taken by making $a = o$, the equation $\varphi(x, y, c_1, c_2, \&c.) = o$ will not be the exact integral of the equation $\varphi'(x, y, dx, dy, d^2y\ \&c.)$. This at first seems to afford an argument against the general accuracy of the method of limits, and a similar circumstance in a case of finite differences has been urged by Lagrange* against the method of limits.

The objection, however, lies not against the method of limits, but against the arbitrary form of the constant quantities. This will be readily understood by considering the instances hereafter adduced.

The last is the case referred to by Lagrange. About the year 1782, M. Charles shewed that an equation to finite differences might have two integrals, each having a constant arbitrary quantity. He applied this reasoning to differential equations and deceived himself as to the result. This result Lagrange gets rid of, by attributing the source of error to the passage from finite to infinitely small quantities.

" Ainsi il faut dire que passage du fini à l'infiniment petit,
" anéantit non-seulement les quantités infiniment petites, mais
" encore la constante arbitraire."

Let us take the equation

$$d^nz + Ad^{n-1}zdx + Bd^{n-2}zdx^2 + \ -\ -\ -\ .Pzdx^n = o \qquad (1)$$

where A, B, C, &c. are constant quantities.

* Séances des écoles normales, &c. 1801, p. 401, &c.

Euler* first gave the integral of this equation, and has since been followed by many other authors.

$1 + Ax + Bx^2 + \&c. \; - - - + Px^n$ is always resolvable, as is well known, into simple or quadratic factors.

The quadratic factors not resolvable into simple factors are, as is also well known of the form

$$(1 - \alpha x)^2 + \beta^2 x^2 \qquad (2)$$

when $\beta = o$, there are two equal simple divisors $(1 - \alpha x)$. now according to Euler, and I believe all authors who have since written on the subject (to the last of whom, Lacroix, I may particularly refer) the part of the integral corresponding to the quadratic factor (2) is

$$c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \qquad (3)$$

where c_1 & c_2 are constant arbitrary quantities, and the part of the integral corresponding to the two equal roots is

$$c_1 e^{\alpha x} + c_2 e^{\alpha x} x \qquad (4)$$

now it would at first view naturally be expected that the expression (4) would be deduced from the expression (3) by taking the limits when $\beta = o$.

But in this case we obtain only $c_1 e^x$ instead of $e^{\alpha x}(c_1 + c_2 x)$ Here the application of limits seems to fail entirely.

The expression (3) is true ***whatever definite*** value we assign to β, and yet is not true of the limit to which this quantity approaches, when β is indefinitely diminished and becomes evanescent.

This, which certainly appears a sort of paradox, may be thus explained:

* Nov. Comm. Acad. Pet. Tom 3.

The expression (3) cannot be considered as including the case of the limit, for the general expression is

$$c_1\, e^{\alpha x} \cos \beta x + c_2 \left(\frac{e^{\alpha x} \sin \beta x}{\beta}\right) \qquad (5)$$

now whatever value we assign to β excepting $\beta = o$, as c_2 is arbitrary $\frac{c_2}{\beta}$ is arbitrary, and therefore may be expressed by c_2. But in order that the expression may be general, it must be retained of the form (5). Then if we make $\beta = o$ it becomes

$e^{\alpha x}\left[c_1 + c_2 \frac{\sin \beta x}{\beta}\right]$ or $c_1 e^{\alpha x} + c_2 e^{\alpha x} x$ because the limit of $\frac{\text{Sin}\, \beta x}{\beta} = x$

I shall not give here the investigation of the form (5) but only mention that I deduced it by a method of integration similar to that which I applied to the differential equation of the Lunar orbit in a preceding paper. By which method I obtain the general integration of Equat. (1) without the intervention of impossible expressions.

Again, if $1 + Ax + Bx^2 + \ldots Px^n$ contain two equal factors of the form $(1-\alpha x)^2 + \beta^2 x^2$ Euler first and lastly Lacroix* have made the corresponding part of the integral

$$e^{\alpha x} \left\{(c_1 + c_4 x) \cos \beta x + (c_2 + c_3 x) \sin \beta x\right. \qquad (6)$$

now when $\beta = o$

this becomes $e^{\alpha x} (c_1 + c_4 x)$, whereas the part of the integral arising from four equal simple factors of the form $(1 - \alpha x)$ is

$e^{\alpha x} (c_1 + c_2 x + c_3 x^2 + c_4 x^4)$ as is well known.

Here then the same difficulty occurs, and the method of limits might be considered as leading in this instance also to an

* Lacroix Traite du Calcul. Diff. &c. tom. 2. p. 319, ed. 2. The first edition has not the case of the equal quadratic factors.

erroneous conclusion. But the form (6) is not sufficiently general to include the case of $\beta = o$. For its general form I find to be by the method above referred to,

$$e^{ax}\left\{ c_1 \cos \beta x + \left(\frac{c_2}{\beta} + \frac{c_4}{2\beta^3}\right)\sin \beta x + \frac{c_3}{2\beta}\, x \sin \beta x - \frac{c_4}{2\beta^2}\, x \cos \beta x \right\} (7)$$

This, because of the arbitrary quantities, agrees with the form (6) in every case but when $\beta = o$.

By making $\beta = o$ in form (7) the result is

$e^{ax}\left\{c_1 + c_2 x + c_3\, x^2 + c_4 x^3\right\}$ because when $\beta = o$

$\frac{\sin \beta x}{\beta} = x$, $\frac{\sin \beta x}{\beta^3} - \frac{x \cos \beta x}{\beta^2} = \frac{x}{1.2}$

The next example is from Lagrange;* it arises from the integration of an equation of finite differences, and which, as that author seems to think, furnishes a strong objection against deducing the properties of differential equations from the limits of finite differences.

Let $\left.\begin{matrix} y, & y', & y'' & \&c. \\ x, & x+i, & x+2\,i & \&c. \end{matrix}\right\}$ be corresponding values

of which the relations are

$$\begin{aligned} y &= a\, x + a^2 \\ y' &= a'\,(x+i) + a' \\ &\&c.\ \&c. \end{aligned} \qquad (8)$$

and $\quad a'\,(x+i) + a' = a\,(x+i) + a^2 \qquad (9)$

Then $y' = a\,(x+i) + a^2$

and $\Delta y = y' - y = a\, i$

Hence $y = \frac{x\,\Delta y}{i} + \frac{(\Delta y)^2}{i^2} \qquad (10)$

* Seances des écoles normales, 1801, p. 401, &c.

of which equation of finite differences the integral is

$$y = a x + a^2$$

The equation (9) is easily reduced to

$$((a' + a) + (x + i)) (a' - a) = o$$

Hence $a - a' = o$

and $a' + a + x + i = o$ (11)

the former equation gives $a = a'$ and therefore

$a' = a''$ &c.

Hence one integral of the equation (10) is

$y = ax + a^2$, where a is any constant arbitrary.

To find the value of a from the latter equation

$a' + a + x + i = o$

Suppose $a = u + m x + n$

m and n being constant, and u a new variable.

Then $a' = u' + m (x + i) + n$

& equation (11) becomes $u' + u + (2 m + 1) x + 2 n + (m + 1) i = o$

If we make $2 m + 1 = o$ and $2 n + (m + 1) i = o$

$$m = -\frac{1}{2} \text{ and } n = -\frac{i}{4}$$

and $u + u' = o$ (12)

with these values of m and n

$a = u - \frac{x}{2} - \frac{i}{4}$ in which value of a, u

may represent any quantity satisfying equation (12)

let $u = b r^x$ (13)

b and r being constant quantities.

Then $u' = b r^{x+i}$ and from equation (12)

$$b r^{x+i} + b^r = o$$

divide by b^r

and $r^i + 1 = o$

therefore $r = (-1)^{\frac{1}{i}}$

and $u = b(-1)^{\frac{x}{i}}$

Hence $a = b(-1)^{\frac{x}{i}} - \frac{x}{2} - \frac{i}{4}$ (14)

where b is a constant arbitrary quantity.

consequently we have obtained two integrals of the equation

$$y = \frac{x'\Delta y}{i} + \frac{(\Delta y)^2}{i^2}$$

viz. $y = ax + a^2$ where a is any constant arbitrary quantity, and $y = ax + a^2$ where

a is any quantity of the form $b(-1)^{\frac{x}{i}} - \frac{x}{2} - \frac{i}{4}$,

b being any constant arbitrary quantity.

If we now suppose i to be diminished, and the limit of $\frac{\Delta y}{i}$ to be taken, *i. e.* $\frac{dy}{dx}$ be substituted, equation (14) becomes

$$y = \frac{x\,dy}{dx} + \frac{(dy)^2}{dx^2} \qquad (16)$$

Now the integral of this equation is as easily appears $y = ax + a^2$ a being a constant arbitrary quantity. This coincides with the first integral of equation (10). The second integral of this equation becomes in this case, because $\frac{i}{4}$ is evanescent and because

$b^2(-1)^{\frac{2x}{i}} = b^2(1)^{\frac{x}{i}} = b^2$

$$y = b^2 - \frac{x^2}{4} \qquad (17)$$

According to this conclusion we have two integral equations each containing an arbitrary quantity for the same differential equation of the first degree. This cannot be. And in fact the equation (17) will not answer. Thus it would appear that the

same reasoning, which applies to the integration of equations of finite differences, when applied to the limits of these equations, viz. to differential equations, leads to error. The above integration is, as to substance, the same as in the work of Lagrange above referred to. He attributes the error to the passage from finite quantities to indefinitely small quantities, and thence to the limits,

But this appears to me a mistaken view of the question: the second value of a (14) is not as general as it ought to be, and cannot apply to the limiting equations, for in equation (13) we assumed $u=br^x$ and then $\acute{u}=br^{x+i}$, now b being any arbitrary constant quantity if we make as in the case of the limits $i=o$, $u=br^x$ and $\acute{u}=br^x$ and therefore, $u=\acute{u}$, but this is impossible for by equat. (12) $u+\acute{u}=o$ an equation that cannot subsist u and $\acute{u}$ being equal and finite quantities. Consequently that the values of u and $\acute{u}$ may apply to the limits, it is necessary that the constant arbitrary quantity should be of the form $i\,b$ instead of b, then when we take the limits by making $i=o$, $u=o$, and $\acute{u}=o$ and therefore $u+\acute{u}=o$.

If we carry on the process with $u=ibr^x$, the second value of

$a=i\,b\,(-1)^{\frac{x}{i}}-\frac{x}{2}-\frac{i}{4}$, b being any constant arbitrary quantity. So that one of the integrals of the equation

$$y=\frac{x\Delta y}{i}+\frac{(\Delta y)^2}{i^2} \text{ is}$$

$$y=ax+a^2$$

where $a=i\,b\,(-1)^{\frac{x}{i}}-\frac{x}{2}-\frac{i}{4}$, b being any constant arbitrary quantity. Here evidently $i\,b$ is arbitrary as to quantity and may be represented by b for all values of i excepting $i=o$.

Hence taking the limits, as before, we have the differential equation.

$$y = \frac{x\,dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

and its integral

$$y = ax + a^2 \text{ becomes, because } a = -\frac{x}{2}$$

$$y = -\frac{x}{2}$$

Integrals of this kind are well known among mathematicians under the name of *particular* integrals.

Thus we have obtained from an equation to finite differences two integrals each containing a constant arbitrary quantity, and then by applying the method of limits we have from thence obtained a differential equation and two integrals, one a common integral, and the other a *particular* one, and thus the method of limits in this, as doubtless in all other cases when properly applied, gives exact results.

The elucidation of these and similar difficulties seems of some importance, when it is considered that Lagrange, to whom unquestionably belongs a very high place among mathematicians, perhaps the highest, appears to have considered the method of limits with less attention, than was due to it, and to have imagined it involved in difficulties that do not belong to it. After having given the preceding example he proceeds to remark on, and make similar objections to, the method of limits, in other instances, and particularly in deducing Taylor's theorem from the equation of finite differences. These objections, as it appears to me, may be easily obviated, and the only proof in every respect unexceptionable of this important Theorem deduced from the limits of the equation of finite differences.